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COUNTING TRIANGLES IN MASSIVE GRAPHS WITH MAPREDUCE[*]

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Abstract. Graphs and networks are used to model interactions in a variety of contexts. There is a growing need to quickly assess the characteristics of a graph in order to understand its underlying structure. Some of the most useful metrics are triangle-based and give a measure of the connectedness of mutual friends. This is often summarized in terms of clustering coefficients, which measure the likelihood that two neighbors of a node are themselves connected. Computing these measures exactly for large-scale networks is prohibitively expensive in both memory and time. However, a recent *wedge-sampling* algorithm has proved successful in efficiently and accurately estimating clustering coefficients. In this paper, we describe how to implement this approach in MapReduce to deal with massive graphs. We show results on publicly available networks, the largest of which is 132M nodes and 4.7B edges, as well as artificially generated networks (using the Graph500 benchmark), the largest of which has 240M nodes and 8.5B edges. We can estimate the clustering coefficient by degree bin (e.g., we use exponential binning) and the number of triangles per bin, as well as the global clustering coefficient and total number of triangles, in an average of 0.33 seconds per million edges plus overhead (approximately 225 seconds total for our configuration). The technique can also be used to study triangle statistics such as the ratio of the highest and lowest degree, and we highlight differences between social and nonsocial networks. To the best of our knowledge, these are the largest triangle-based graph computations published to date.

Key words. triangle counting, clustering coefficient, triangle characteristics, large-scale networks, MapReduce

AMS subject classifications. 05C85, 62H12, 65Y10, 68W15, 68W20

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1. Introduction.

Over the last decade, graphs have emerged as the standard for modeling interactions between entities in a wide variety of applications. Graphs are used to model infrastructure networks, the World Wide Web, computer traffic, molecular interactions, ecological systems, epidemics, coauthors, citations, and social interactions, among others. Understanding the frequency of small subgraphs has been an important aspect of graph analysis.

Despite the differences in the motivating applications, some topological structures have emerged to be important across all these domains. The most important such subgraph is the triangle (3-clique). Many networks, especially social networks, are known to have many triangles. This is thought to be because social interactions exhibit homophily (people befriend similar people) and transitivity (friends of friends become friends). The notion of *clustering coefficient* is inspired by this observation and is the standard method of summarizing triangle counts [61, 40]. It is well known that some networks, especially social networks, have much higher clustering coefficients

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than random networks [38, 39, 41]. Triangle measures are important for understanding network structure and evolution [24, 48, 20, 60] and reproducing the degreewise clustering coefficients of a network is important for generative models [22, 48, 51].

1.1. Our contributions. For large graphs, computing triangle-based measures can be expensive. The standard approach is to find all wedges, i.e., paths of length 2, and check to see if they are closed, i.e., the edge that completes the triangle exists. Previous work presents a wedge-sampling approach for approximating clustering coefficients [50, 53]; this is in contrast to sampling single edges, which is a more obvious but less reliable technique. In [53], it is shown that the wedge-sampling approach is orders of magnitude faster than enumeration and is both faster and has less variance than edge-sampling techniques.

In this paper, we show that the wedge-sampling approach scales to massive networks using MapReduce, a framework well suited to sampling. Previous distributed triangle counting algorithms have had to deal with problems of finding triangles where edges are stored on different processors and skewed degree distributions lead to load balancing issues [54, 3]. In contrast, our wedge-sampling approach in MapReduce deals with these issues seamlessly and recommends sampling as a general technique for large graphs since it leads to fast serial algorithms as well as scalable parallel implementations. We describe our contributions in more detail as follows:

- We present a parallelization of our wedge-based sampling algorithm in the MapReduce framework. The premise of wedge sampling is to set up a distribution on the vertices (as potential wedge centers) and use that to sample the actual wedges. Designing a serial algorithm is easier, since the information to compute the distribution and then form the wedges is all local. In the MapReduce implementation, edges are distributed arbitrarily; therefore, it takes several passes to compute the necessary distribution, create the sample wedges, and finally check if they are closed.
- Additionally, we show that MapReduce enables computing multiple clustering coefficients for the same graph (e.g., binned by degree) for essentially the same cost as computing the single global clustering coefficient. Since the clustering coefficient generally varies with degree, it is helpful to see the profile versus a single value because these profiles are useful in graph characterization and modeling.
- We give extensive experimental results on both real-world networks from the Laboratory for Web Algorithms as well as artificial networks created according to the Graph500 benchmark. We have multiple examples with more than a billion edges. To the best of our knowledge, these are the largest triangle computations to date.
- Results demonstrate the efficiency of our algorithm. For instance, we estimate the cost of computing clustering coefficients per (logarithmically) binned degree to be an average of 0.33 seconds per million edges plus overhead, which is approximately 225 seconds total for our 32-node Hadoop cluster. Hence, a graph with over 9B edges requires less than one hour of computation. Note that the global clustering coefficient and total triangles are also computed.
- A straightforward implementation requires that the entire edge list be "shuffled" three times. We show how to greatly reduce the shuffle volume with some clever implementation strategies that are able to filter the edge list during the "map" phase. We discuss the implementation details and show comparisons of performance.
- A feature of wedge-based sampling is that the closed wedges are uniform random triangles. Hence, we also give experimental results characterizing triangles in terms of their minimum and maximum degrees. Triangles from social networks tend to be somewhat assortative, whereas triangles from other types of networks are not.

1.2. Related work on triangle counting. Enumeration algorithms for finding triangles are either node- or edge-centric. Node-centric algorithms iterate over all nodes and, for each node v, check all pairs among the neighbors of v for being connected. Edge-centric algorithms, on the other hand, go over all edges (u, v) and seek common neighbors of u and v. Chiba and Nishizeki [13] proposed a node-centric algorithm that orders the vertices by degree and processes each edge only once, by its lower-degree vertex. They showed that this algorithm runs in $O(m\alpha(G))$-time, where m is the number of edges and $\alpha(G)$ is the arboricity of the graph G. (Arboricity is defined as the minimum number of forests into which its edges can be partitioned and can be considered as a measure of how dense the graph is.) Schank and Wagner [50] used the same idea for their *forward* algorithm. Chu and Cheng studied an I/O efficient implementation of the same algorithm [15]. Latapy proved that the forward algorithm runs in $O(m^{3/2})$-time and proposed improvements that reduce the search space [31]. Latapy also showed that the runtime of this algorithm becomes $O(mn^{1/\alpha})$ for graphs with power-law degree distributions, where α is the power-law coefficient and n is the number of vertices [31]. Arifuzzaman, Khan, and Marathe [3] give a massively parallel algorithm for computing clustering coefficients. Pearce, Gokhale, and Amato [45] used triangle counting as an application to show the effectiveness of external memory algorithms for massive graph analysis.

Enumeration algorithms, however, can be expensive, due to the extremely large number of triangles (see, e.g., Table 2), even for graphs of moderate size (millions of vertices). Much theoretical work has been done on characterizing the hardness of exhausting triangle enumeration and finding weighted triangles [44, 62]. Eigenvalue/trace-based methods adopted by Tsourakakis [58] and Avron [4] compute estimates of the total and per-degree number of triangles. However, the compute-intensive nature of eigenvalue computations (even just a few of the largest magnitude) makes these methods intractable on large graphs.

Most relevant to our work are sampling mechanisms. Tsourakakis et al. [56] initiated the sparsification methods, the most important of which is Doulion [59]. This method sparsifies the graph by retaining each edge with probability p, counts the triangles in the sparsified graph, and multiplies this count by p^{-3} to predict the number of triangles in the original graph. One of the main benefits of Doulion is its ability to reduce large graphs to smaller ones that can be loaded into memory. However, the estimates can suffer from high variance [64]. Theoretical analyses of this algorithm (and its variants) have been the subject of various studies [29, 57, 42]. Another sampling approach has been proposed by Kolountzakis et al. [29], which involves both edge and triple-node sampling (a generalization of wedge sampling). A MapReduce implementation of their method could potentially use many of the same techniques presented here. Alternative sampling mechanisms have been proposed for streaming and semistreaming algorithms [5, 27, 7, 12]. Most recently, Jha, Seshadhri, and Pinar [25] showed how wedge sampling can be performed when the graph is observed as a stream of edges and generalized their method for graphs with repeated edges [26]. An alternative approach and its parallelization were proposed by Tangwongsan, Pavan, and Tirthapura [55]. Many of these sampling procedures are by their nature quite amenable to a MapReduce implementation.

The wedge-sampling approach used in this paper, first discussed by Schank and Wagner [49], is a sampling approach with the high accuracy and speed advantages of other sampling-based methods (like Doulion) but a hard bound on the variance. Previous work by a subset of the authors of this paper [53] presents a detailed empirical study of wedge sampling. It was also shown that wedge sampling can compute a variety of triangle-based metrics including degreewise clustering coefficients and uniform randomly sampled triangles. This distinguishes wedge sampling from previous sampling methods that can only estimate the total number of triangles.

1.3. Related work on MapReduce for graph analytics. MapReduce [17] is a conceptual programming model for processing massive data sets. The most popular implementation is the open-source Apache Hadoop [1] along with the Apache Hadoop Distributed File System (HDFS) [1], which we have used in our experiments. MapReduce assumes that the data is distributed across storage in roughly equal-sized blocks. The MapReduce paradigm divides a parallel program into two parts: a *map* step and a *reduce* step. During the map step, each block of data is assigned to a *mapper* which processes the data block to emit key-value pairs. The mappers run in parallel and are ideally local to the block of data being processed, minimizing communication overhead. Between the map and reduce steps, a parallel *shuffle* takes place in order to group all values for each key together. This step is hidden from the user and is extremely efficient. For every key, its values are grouped together and sent to a *reducer*, which processes the values for a single key and writes the result to file. All keys are processed in parallel.

MapReduce has been used for network and graph analysis in a variety of contexts. It is a natural choice, if for no other reason than the fact that it is widely deployed [34]. Pegasus [28] is a general library for large-scale graph processing; the largest graph Kang, Tsourkakis, and Faloutsos considered was 1.4M vertices and 6.6M edges using the PageRank analytic, but they did not report execution times. Lin and Schatz [36] propose some special techniques for graph algorithms such as PageRank that depend on matrix-vector products. MapReduce sampling-based techniques that reduce the overall graph size are discussed by Lattanzi et al. [32].

In terms of triangle counting and computing clustering coefficients, Cohen [16] considers several different analytics including triangle and rectangle enumeration. Plantenga [47] has studied subgraph isomorphism (i.e., finding small graph patterns such as triangles), including Cohen's algorithm as a special case. (We use Plantenga's implementation of Cohen's triangle enumeration algorithm for comparison in our subsequent numerical results.) For a nontriangle pattern, Plantenga's SGI code ran on a 7.6B vertex graph with 107B undirected edges in 620 minutes on a 64-node Hadoop cluster. Wu et al. [63] have also studied triangle enumeration using MapReduce with running times of roughly 175 seconds on a graph with 1.6M nodes and 5.7M edges. Suri and Vassilvitskii [54] proposed a MapReduce implementation for exact per-node clustering coefficients. Most naive partitioning schemes do not give efficient parallelization because of high-degree vertices, and their result involves new partitioning methods to avoid this problem. We discuss how both [47] and [54] compare to our method in section 5.3. SAHAD [66] has a Hadoop program that uses sampling techniques based on graph coloring to find subgraphs, but it is limited to tree patterns. Ugander et al. [60] analyzed the Facebook graph with 721M nodes and 69B edges (representing friendships) on a 2,250 node Hadoop cluster. They sampled 500,000 nodes and computed the exact local clustering coefficient for each sampled node. They reported (binned) averages of the local clustering coefficients. We note that the binned local clustering coefficient is different from the binned global clustering coefficient which we calculate in this paper. Additionally, the approach of computing the exact local clustering coefficient for a set of sample nodes is nontrivial for general graphs since assembling the neighbors for high-degree nodes is extremely expensive. In the case of the Facebook graph, the maximum number of neighbors is only 5,000 (per Facebook policy); compare to our graphs which have nodes with over 1 million neighbors.

2. Background.

2.1. Global clustering coefficient. Let $G = (V, E)$ be an undirected graph with $n = |V|$ nodes and $m = |E|$ undirected edges. We assume the vertices are indexed by $i = 1, \ldots, n$. Let d_i denote the degree of vertex i; degree-zero vertices are ignored. A *wedge* is a length-2 path. Let p_i denote the number of wedges centered at vertex i; i.e., $p_i = \binom{d_i}{2} = \frac{d_i(d_i-1)}{2}$. A wedge is *closed* if its endpoints are connected and *open* otherwise. The *center* of a wedge is the middle vertex. A *triangle* is a cycle with three vertices. A closed wedge forms a triangle; conversely, a triangle corresponds to *three* closed wedges. Let t_i denote the number of triangles containing node i, which is equal to the number of closed wedges centered at node i. The *node-level clustering coefficient* (first used in [61]) is

$$c_i = \frac{t_i}{p_i} = \frac{\text{number of triangles incident to node } i}{\text{number of wedges centered at node } i}.$$

Thus, c_i measures how tightly the neighbors of a vertex are connected among themselves.

We define W to be the set of all wedges in G and $p = |W| = \sum_i p_i$. We partition W into two disjoint subsets as follows:

$$W_0 = \{w \in W | w \text{ open}\},$$
$$W_3 = \{w \in W | w \text{ closed}\}.$$

The subscript of 3 for the closed wedges indicates that each triangle creates three wedges in W_3. Let $t = \frac{1}{3}\sum_i t_i = \frac{1}{3}|W_3|$ denote the total number of triangles (since each triangle is counted thrice). The *(global) clustering coefficient* (also known as the transitivity) [40] of an undirected graph is given by

$$c = \frac{|W_3|}{|W|} = \frac{\sum t_i}{\sum p_i} = \frac{3t}{p} = \frac{3 \times \text{total number of triangles}}{\text{total number of wedges}}. \tag{2.1}$$

At the global level, c is an indicator of how tightly nodes of the graph are connected.

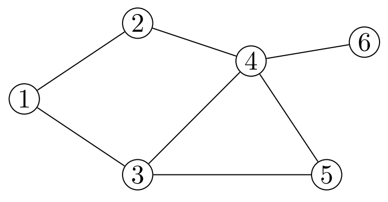
2.2. Binned degreewise clustering coefficient. In this paper, we will be using the binned degreewise clustering coefficients, which measure how tightly the neighborhood of vertices of a specified degree group are connected. Let $D \subseteq \{ d_i, d_j, \ldots \}$ be a subset of degrees. (Recall that we ignore degree-zero nodes.) We define $V_D = \{ i \in V \mid d_i \in D \}$ and $n_D = |V_D|$. In many cases, we are interested in a single degree, i.e., if $D = \{ d \}$, then V_d is the set of nodes of degree d and n_d is the number of nodes of degree d.

We define W_D to be the set of all wedges centered at a node in V_D and p_D to be the total number of wedges centered at nodes in V_D, i.e., $p_D = |W_D|$. If $D = \{ d \}$, then $p_d = n_d\binom{d}{2}$. We partition the set W_D into four disjoint subsets as follows:

$$W_{D,0} = \{ w \in W_D \mid w \text{ open} \},$$
$$W_{D,q} = \{ w \in W_D \mid w \text{ closed and has } q \text{ nodes in } V_D \} \quad \text{for } q = 1, 2, 3.$$

Define $p_{D,q} = |W_{D,q}|$ for $q = 0, 1, 2, 3$. Since $p_D = \sum_q p_{D,q}$, we can define *binned degreewise clustering coefficient*, c_D, as the fraction of closed wedges in W_D; i.e.,

$$c_D = (p_{D,1} + p_{D,2} + p_{D,3})/p. \tag{2.2}$$



$n = 6,\ m = 7,\ \{d_i\} = \{2, 2, 3, 4, 2, 1\}$
$p = 12,\ \{p_i\} = \{1, 1, 3, 6, 1, 0\}$
$t = 1,\ \{t_i\} = \{0, 0, 1, 1, 1, 0\}$
$c = 0.25,\ \{c_i\} = \{0, 0, 1/3, 1/6, 1, 0\}$
$\{n_d\} = \{1, 3, 1, 1\},\ \{p_d\} = \{0, 3, 3, 6\}$
$\{t_d\} = \{0, 1, 1, 1\},\ \{c_d\} = \{0, 1/3, 1/3, 1/6\}$

FIG. 1. *Example graph with various quantities highlighted.*

The formula for triangles is more complex and given by

$$t_D = p_{D,1} + \frac{1}{2} \cdot p_{D,2} + \frac{1}{3} \cdot p_{D,3},$$

since for each triangle there is either one wedge in $W_{D,1}$, two wedges in $W_{D,2}$, or three wedges in $W_{D,3}$. Figure 1 shows examples of these quantities when the bins are all singletons: $\{1\}, \{2\}, \{3\}, \{4\}$.

3. Wedge sampling for triadic measures.

For a more detailed exposition of wedge sampling and empirical tests of its behavior, we refer the reader to [53]. For completeness, we review the relevant concepts and calculations here.

3.1. Hoeffding's inequality.

The following result is a simple corollary of Hoeffding's inequality [23] (refer to Theorem 1.5 in [18]); the proof can be found in [53]. We say that ε is the *error* and $(1 - \delta)$ is the *confidence*.

THEOREM 3.1. *Let $X_1, X_2, \dots, X_k$ be independent random variables with $0 \le X_i \le 1$ for all $i = 1, \dots, k$. Define $\bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$. Let $\mu = \mathbb{E}[\bar{X}]$. For any positive ε, δ, setting $k \ge \lceil 0.5\varepsilon^{-2}\ln(2/\delta)\rceil$ yields*

$$\text{Prob}\left\{|\bar{X} - \mu| \ge \varepsilon\right\} \le \delta.$$

3.2. Binned degreewise clustering coefficients and triangles.

The strategy for computing the clustering coefficient per degree (or degree range) is similar to that described for the degreewise clustering coefficient in [53].

THEOREM 3.2 (binned degreewise clustering coefficient). *For $\varepsilon, \delta > 0$, set $k \ge \lceil 0.5\,\varepsilon^{-2}\ln(2/\delta)\rceil$. For $i = 1, \dots, k$, choose wedge w_i uniformly at random (with replacement) from W_D and let X_i be defined as*

$$X_i = \begin{cases} 1 & \textit{if } w_i \textit{ is closed,} \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\text{Prob}\left\{|\hat{c}_D - c_D| \ge \varepsilon\right\} \le \delta \quad \textit{for} \quad \hat{c}_D = \frac{1}{k}\sum_{i=1}^{k} X_i.$$

Proof. Observe that $c_D = \mathbb{E}[\bar{X}]$ since it is the probability that a random wedge in W_D is closed. The proof follows immediately from Theorem 3.1. ∎

Choosing uniform random wedges. We do not want to form all wedges explicitly. Instead, we *implicitly* generate random wedges. Observe that the number of wedges centered at vertex i is exactly $\binom{d_i}{2}$, and $p = \sum_i \binom{d_i}{2}$. That leads to the following procedure. To select a random wedge, recall that $p_D = |W_D|$. Therefore, first choose vertex $i \in V_D$ with probability $\binom{d_i}{2}/p_D$. Second, choose two distinct neighbors of vertex i to form a random wedge. To set up this distribution, we need to compute the degree distribution. If $D = \{\, d\,\}$ (a singleton), then all nodes in V_d are equally probable. If $D = \{\, 0, +\infty\,\}$, then the weight of vertex i is $\binom{d_i}{2}/p$.

Estimating the number of triangles is slightly more complicated since each closed wedge may have one, two, or three vertices in V_D.

THEOREM 3.3 (degreewise triangle count [53]). *Let the conditions of Theorem* 3.2 *hold. For each w_i, let Y_i be defined as*

$$Y_i = \begin{cases} 1 & \textit{if } w \in W_{D,1}, \\ \frac{1}{2} & \textit{if } w \in W_{D,2}, \\ \frac{1}{3} & \textit{if } w \in W_{D,3}, \\ 0 & \textit{if } w \in W_{D,0} \textit{ (open).} \end{cases}$$

Then

$$\text{Prob}\left\{|\hat{t}_D - t_D| \geq \varepsilon \cdot p_D\right\} \leq \delta \quad \textit{for} \quad \hat{t} = p_D \cdot \frac{1}{k}\sum_{i=1}^{k} Y_i.$$

Proof. We claim $\mathbb{E}[Y] = t_D$. Suppose that w is selected from W_D uniformly at random. Observe that

$$\begin{aligned} \mathbb{E}[Y] &= \text{Prob}\{w \in W_{D,1}\} + \frac{\text{Prob}\{w \in W_{D,2}\}}{2} + \frac{\text{Prob}\{w \in W_{D,3}\}}{3} \\ &= 1 \cdot \frac{p_{D,1}}{p_D} + \frac{1}{2} \cdot \frac{p_{D,2}}{p_D} + \frac{1}{3} \cdot \frac{p_{D,3}}{p_D} \\ &= t_D/p_D \end{aligned}$$

per (2.2). Hence, from Theorem 3.1 we have

$$\text{Prob}\left\{|\hat{t}_D/p_D - t_D/p_D| \geq \varepsilon\right\} \leq \delta,$$

and the theorem follows by multiplying the inequality by p_D. ∎

3.3. Computing a random sample of the triangles. In addition to knowing the number of triangles in a graph, it may also be interesting to consider the properties of those triangles. For instance, Durak et al. [19] consider the differences in node degrees in a triangle.

It turns out that the closed wedges discovered during the wedge sampling procedure are triangles sampled uniformly (with replacements). Hence, we can study these randomly sampled triangles to estimate the overall characteristics of triangles in the graph.

THEOREM 3.4. *Let W_s be a random sample of the wedges of a graph G, and let $T_s \subseteq W_s$ triangles that are formed by the closed wedges in W_s. Then each triangle in T_s is a uniform random sample from the triangles of G.*

Proof. The proof depends on observing that a triangle being chosen depends only on one of its three wedges being chosen. Since the wedge sample is uniformly random, each triangle is equally likely to be picked, and there is no dependency between any pair of triangles, which implies a uniform sample. ∎

3.4. Practical performance of wedge sampling. Earlier work by a subset of the authors [53] provides a thorough study on how the techniques described above perform in practice. As expected, tremendous improvements are achieved in runtimes compared to full enumeration, especially for large graphs, since the number of samples is independent of graph size. Specifically, we see speed-ups of more than 1000X with errors in the clustering coefficient of less than 0.002. Additionally, in comparison to the Doulion method (an edge-sampling technique) we obtain speed-ups of 5X or more while obtaining the same accuracy. The ability to adapt our wedge-sampling method to computing binned degreewise clustering coefficients and triangle sampling is also a benefit in comparison to edge-based sampling.

Our goal in this work is to implement the wedge sampling approach within the MapReduce framework and provide evidence that it can scale to much larger problems.

4. MapReduce implementation.

4.1. Overview. We now present a MapReduce algorithm for estimating the clustering coefficients and number of triangles in a graph. For details on MapReduce, see Lin and Dyer [35]; we have emulated their style in our algorithm presentations. We use the open-source Hadoop implementation of MapReduce and HDFS for storing data. Each MapReduce job takes one or more distributed files as input. These files are automatically stored as *splits* (also known as blocks), and one mapper is launched per split. The mappers produce key-value pairs. All values with the same key are sent to the same reducer. The number of reducers is specified by the user. Each MapReduce job produces a single HDFS output file. A MapReduce job accepts *configuration parameters*, which are passed along as data to the mapper and reducer functions; we discuss these in more detail in the sections that follow. The set of MapReduce jobs in our algorithm is coordinated by a Hadoop Java program running on a single *client node*.

In our code, we assume the nodes are binned by degree as discussed in section 4.2. Computation of the global clustering coefficient is a special case which can be computed by either looking only at a single bin containing all degrees or using a weighted average of the binned clustering coefficients (see section 4.6.2).

Our input is an undirected edge list where the node identifiers are 64-bit integers; we assume no duplicates or self-edges and no particular ordering. We divide our MapReduce algorithm into three *major phases* plus postprocessing, as presented in Figure 2. Each major phase makes a complete pass through the edge list. The first phase sets up the distribution on wedges. The second phase creates the sample wedges. Finally, the third phase checks whether the sample wedges are closed. In all three phases, we have strategies to reduce the data volume in the shuffle phase (between the map and reduce), discussed in detail in the sections that follow.

4.2. Binning. We define degree bins in a parameterized way as follows. Let τ be the number of singleton bins, and let $\omega > 1$ be the rate of growth on the bin sizes. The first τ bins are singletons containing degrees $1, 2, \ldots, \tau$, respectively. The remaining bins grow exponentially in size.

We describe the lowest degree of bin k as

$$\textsc{BinLoDeg}(k) = \begin{cases} k & \text{if } k \leq \tau, \\ \lceil (\omega^{(k-\tau)} - 1)/(\omega - 1) \rceil + \tau & \text{otherwise.} \end{cases} \tag{4.1}$$

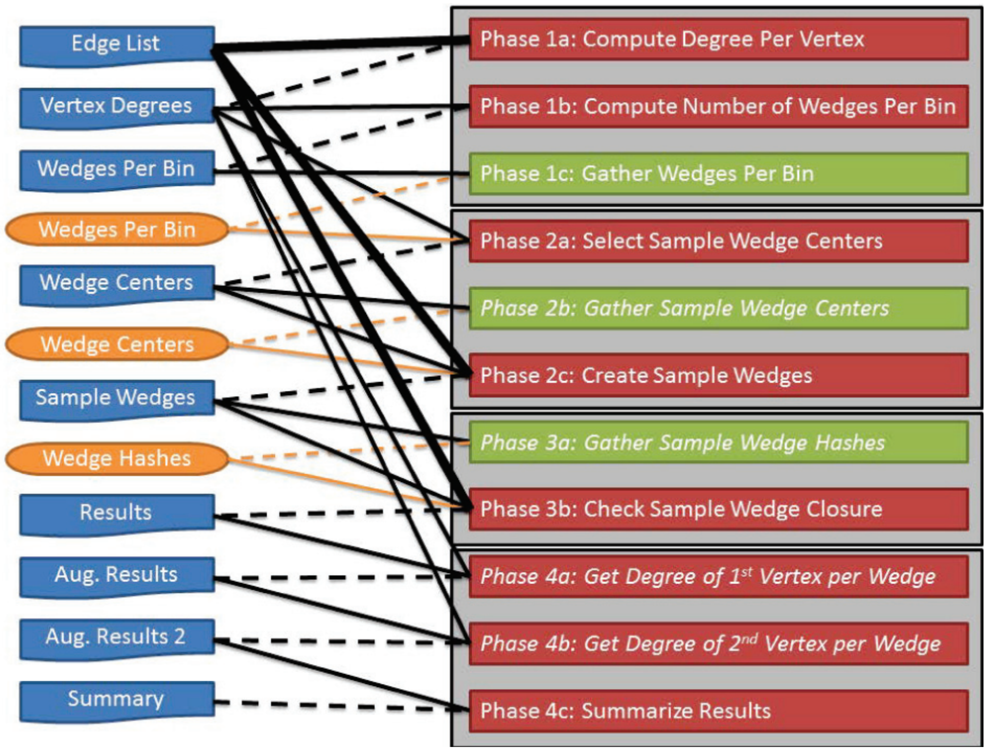

Edge List
Vertex Degrees
Wedges Per Bin
Wedges Per Bin
Wedge Centers
Wedge Centers
Sample Wedges
Wedge Hashes
Results
Aug. Results
Aug. Results 2
Summary
Phase 1a: Compute Degree Per Vertex
Phase 1b: Compute Number of Wedges Per Bin
Phase 1c: Gather Wedges Per Bin
Phase 2a: Select Sample Wedge Centers
Phase 2b: Gather Sample Wedge Centers
Phase 2c: Create Sample Wedges
Phase 3a: Gather Sample Wedge Hashes
Phase 3b: Check Sample Wedge Closure
Phase 4a: Get Degree of 1st Vertex per Wedge
Phase 4b: Get Degree of 2nd Vertex per Wedge
Phase 4c: Summarize Results

FIG. 2. *Algorithm overview for estimating clustering coefficients and counting triangles, both binned by degree. Red boxes indicate a MapReduce job, while green represents a serial operation on the client node. Blue boxes indicate data files. The edge list is provided by the user; all other data files are produced by the method. Orange boxes indicate data that is passed as a "configuration parameter" to all mappers. Solid lines indicate consumption of data. Dotted lines indicate creation of data.*

The highest degree for bin k is just one less than the lowest degree of bin $k+1$. For a given degree d, we can easily look up its bin as

$$\textsc{BinId}(d) = \begin{cases} d & \text{if } d \leq \tau, \\ \lfloor \log(1 + (\omega - 1)(d - \tau)) / \log(\omega) \rfloor + \tau & \text{otherwise.} \end{cases} \tag{4.2}$$

In our implementation, τ and ω are communicated to each MapReduce job as configuration parameters.

For $\tau = 2$ and $\omega = 2$, the bins are $\{1\}$, $\{2\}$, $\{3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, \ldots, 16\}$, $\{17, \ldots, 32\}$, and so on. Note that bin $\{1\}$ cannot have any wedges, so we just ignore it. Let $\bar{d}$ be an upper bound on the highest degree for a given graph. Then choosing $\tau = 1$ and $\omega = \bar{d}$ yields bins $\{1\}, \{2, \ldots, \bar{d}\}$. In other words, we have a single bin containing all vertices (excepting degree-1 vertices). On the other hand, choosing $\tau = \bar{d}$ yields $\{1\}, \{2\}, \ldots, \{\bar{d}\}$. Here, every bin is a singleton.

We are not constrained to (4.2) for computing the bins; we can use any procedure such that each degree is assigned to a single bin. Likewise, (4.1) is optional and used to reduce the shuffle volume in Phase 2c.

ALGORITHM 1. Compute degree per vertex (Phase 1a).

```
method MAP($v, w$)                                    ▷ Input is **edge list file**
    EMIT($v, 1$)
    EMIT($w, 1$)                                      ▷ Emit both for an undirected graph

method REDUCE($v, \{ x_1, x_2, \dots \}$)
    $d \leftarrow$ SUM($\{ x_1, x_2, \dots \}$)                  ▷ Compute degree
    EMIT($v, d$)                                      ▷ Output is **vertex degree file**
```

ALGORITHM 2. Compute number of wedges per bin (Phase 1b).

```
parameters: $\tau, \omega$                                   ▷ Binning parameters

method MAP($v, d$)                                    ▷ Input is **vertex degree file**
    $b \leftarrow$ BINID($d, \tau, \omega$)                        ▷ Compute bin ID
    $n \leftarrow 1$                                             ▷ Number of vertices
    $p \leftarrow d \cdot (d-1)/2$                                ▷ Number of wedges
    EMIT($b, (n, p)$)

method REDUCE($b, \{ (n_1, p_1), (n_2, p_2), \dots \}$)       ▷ One reduce function per bin
    $n \leftarrow$ SUM($\{ n_1, n_2, \dots \}$)                   ▷ Number of vertices in bin
    $p \leftarrow$ SUM($\{ p_1, p_2, \dots \}$)                   ▷ Number of wedges in bin
    EMIT($b, n, p$)                                   ▷ Output is **wedges per bin file**
```

4.3. Phase 1: Compute degree-based statistics.

4.3.1. Phase 1a: Compute degree per vertex. Phase 1a is a straightforward MapReduce task—computing the degree of each vertex. The MAP and REDUCE functions are described in Algorithm 1. The input is the *edge list file*; each entry is a pair of vertex IDs (v, w) that define an edge. The MAP function is called for each edge (v, w) and emits two key-value pairs keyed to the vertex IDs and having a value of 1. The REDUCE function gathers all the values for each vertex and sums them to determine the degree. The final output to HDFS is a *vertex degree file*; each entry is of the form (v, d), where v is a vertex ID and d is its degree.

Algorithm 1 shows a simple version of the code. To make the code more efficient, we collect local counts within each mapper (using a Java `Map` container) and emit the totals. This technique is called an in-memory combiner [36]. We found the in-memory combiner to reduce shuffle volume more than employing the reducer as a combiner.

4.3.2. Phase 1b: Compute number of wedges per bin. Phase 1b works with the output of Phase 1a (*vertex degree file*) to compute the number of wedges per bin. The MAP and REDUCE functions for Phase 1b are presented in Algorithm 2. The input is the list of degrees per vertex. The MAP function is called for each vertex (with its associated degree) and emits the number of wedges for that vertex, keyed to the appropriate bin. The REDUCE function simply combines the results for each bin. The final output is a *wedges per bin file*; each entry is of the form (b, n_b, p_b), where b is the bin ID, n_b is the number of vertices in the bin, and p_b is the number of wedges in the bin.

Algorithm 3. Determine number of samples per vertex (Phase 2a).

```
parameter: $k$                                        ▷ Desired number of samples per bin
parameter: $\theta$                                   ▷ Represents wedges per bin object

method Map($v, d$)                                    ▷ Input is vertex degree file
    $b \leftarrow$ BinId($d$)                         ▷ Compute bin ID
    $p \leftarrow \theta(b)$                          ▷ Total number of wedges in bin containing $v$
    $q^* \leftarrow (d \cdot (d-1)/2) \cdot k/p$      ▷ Ideal number of samples, likely noninteger
    $x \leftarrow$ rand ($[0,1]$)                     ▷ Uniform random number in $[0,1]$
    $q \leftarrow \{\, x \geq (q^* - \lfloor q^* \rfloor) \,\}\ ?\ \lceil q^* \rceil : \lfloor q^* \rfloor$ ▷ Number of sample wedges centered at $v$
    if $q \geq 1$ then                                ▷ Skip vertices with no samples
        Emit($v, d, q, p$)                            ▷ Output is wedge centers file
    end if
```

Once again, we have shown a simple version of the algorithm in Algorithm 2. To make the code more efficient, we collect local counts within each mapper (using a Java `Map` container) and emit the totals.

For the case of a single bin, strictly speaking, Phase 1b is unnecessary. Instead, we could have used a Hadoop *global counter* to tally the total wedges in the reduce step of Phase 1a.

4.3.3. Phase 1c: Gather wedges per bin. From the *wedges per bin file* (output of Phase 1b), we create a *wedges per bin object*, which acts as a function θ such that $\theta(b)$ is the number of wedges in bin b. The work is performed entirely in our main program running on the client node. It reads Phase 1b output from HDFS, stores wedges per bin values in a Java `Map` container, and launches the next MapReduce job (Phase 2a), sending the *serialized* container as a configuration parameter.

4.4. Phase 2: Select wedge samples.

4.4.1. Phase 2a: Select sample wedge centers. The input to Phase 2a is the *vertex degree file* along with the *wedges per bin object*, which is passed as a configuration parameter. Phase 2a calculates the number of sample wedges centered at each vertex. The Map function is shown in Algorithm 3. The Map function is called for each (vertex ID, degree) pair. From this, we can calculate the expected number of wedges that would be sampled from the vertex for a uniform random sample, q^*. This number is unlikely to be integral. Rounding up would produce far too many wedges. Instead, we use probabilistic rounding. For instance, if $q^* = 0.1$, then there is a 10% change of producing $q = 1$ wedges and a 90% chance of producing no wedges, $q = 0$. We are only off by at most one, so if $q^* = 1.1$, then there is a 10% change of producing $q = 2$ wedges and a 90% chance of producing $q = 1$ wedge. Hence, the expected number of wedges for this vertex is exactly q^*. Only vertices with at least one sample wedge are emitted. The final output is a *wedge centers file*; each entry is of the form (v, d, q, p), where v is the vertex ID, d is the vertex degree, q is the number of sample wedges centered at that vertex, and p is the total number of wedges in the bin containing v. The Reduce function is just the identity map and is not shown.

4.4.2. Phase 2b: Gather sample wedge centers. Phase 2b is an optional step that generates a Java `Map` of wedge centers and their bin IDs based on the output of Phase 2a (*wedge centers file*). We represent this object as a function γ such that

$$\gamma(v) = \begin{cases} 0 & \text{if Phase 2b is skipped,} \\ 1 & \text{if } v \text{ is not a wedge center,} \\ b \geq 2 & \text{if } v \text{ is a wedge center, in which case } b \text{ is the bin ID.} \end{cases}$$

This *wedge centers object* has one value for every vertex appearing in a wedge center. It is serialized and passed as a configuration parameter to Phase 2c, where it is used to filter the edges that are emitted by the MAP function.

Note that Hadoop imposes a limit on the size of the configuration parameters (5MB by default). If the number of wedge centers is too large (a few hundred thousand samples will exceed 5MB), then other options must be explored. One alternative is to pass the container to the MAP tasks using the Hadoop distributed cache; however, we have not implemented this idea.

Phase 2b is optional and can be skipped if there are too many wedge centers. We demonstrate the benefits of this step in section 5.

4.4.3. Phase 2c: Create sample wedges. In Phase 2c, the goal is to take each sample wedge center (from the *wedge center file*), collect its neighbors (from the *edge list file*), and create a set of sample wedges. We merge each vertex and its neighbors at the reduce phase. If it exists, the optional *wedge center object* is used to filter the edges that are shuffled, ignoring all edges that are not adjacent to a sampled wedge center. The algorithm is shown in Algorithm 4. For clarity, we give a separate MAP function for each input type. In the actual implementation, we have to determine the input type on the fly, because both input files are of Hadoop type `Text`. For input from the *wedge centers file*, the MAP function simply passes along its degree and sample wedge count (i.e., the number of wedges to be sampled from the vertex).

For input from the *edge list file*, the MAP function checks to see if the edge is adjacent to a wedge center. If so, it is passed based on the outcome of a random coin flip. The aim of the REDUCE phase is to generate random wedges centered at a vertex (say, v). The most naïve MAP implementation would forward all edges incident to v, so that wedges can be selected from them. A major problem with this is that if the number of samples k is much less than the degree of v, most of the communication is unnecessary. For example, the highest degree vertices of a social network graph might link to millions of edges, but k is in the tens of thousands or less; therefore, most of the incident edges will not participate in sampled wedges centered at these vertices. We have a probabilistic fix to address this situation.

We do not have the vertex degree readily available, but we do know its bin and therefore a lower bound on its degree. Consider a vertex v of degree greater than $d_{\min}$, where $2k \leq d_{\min}/2$. We send just some of the incident edges to v, with independent probability $\phi = 4k/d_{\min} \leq 1$. Then the expected number of edges to send is $4k(d_v/d_{\min})$. Note that this expectation is at least $4k$. Getting less than $2k$ edges is potentially disastrous, but the probability of this is minuscule. By a multiplicative Chernoff bound (given below), the probability of such an event is $\exp(-k/8)$. For $k = 1000$ (a tiny sample size), the probability is less than 10^{-55}.

THEOREM 4.1 (multiplicative Chernoff bound [18]). *Let* $X = \sum_{i \leq r} X_i$, *where each* X_i *is independently distributed in* $[0, 1]$. *Then*

$$\text{Prob}\{X \leq (1-\delta)\mathbb{E}[X]\} \leq \exp(-\delta^2 \mathbb{E}[X]/2).$$

If d_v is not too far from $d_{\min}$, then the expectation $4k(d_v/d_{\min})$ is potentially much smaller than d_v. Hence, we get the desired number of random edges without sending too many.

Even with this improvement, the data passed forward may be too large to fit into the reducer's memory. We use a feature of Hadoop called *secondary sort* to ensure that the data arrives presorted. Note that the key used for passing along the vertex information is $v{:}0$ and the key for the edges is of the form $v{:}y$, where y is a random positive integer. This data is all mapped to the key v, but the values following the colon control the sort of the values associated with v. The secondary key of zero ensures that the degree and wedge count data are first. The secondary keys for edges (y) ensure that the adjacent edges are randomly sorted; otherwise, Hadoop would present the edges in their order of arrival, which could bias the selection.

From the secondary sort, the wedge center must be first in the values list at the reduce phase, if it exists. If it does not exist, then there is nothing to do. Recall that for each wedge center v, we have its degree, d, and a desired number of wedge samples, q. Each wedge must be randomly sampled *with* replacement. The two edges of a single wedge are sampled *without* replacement. So, wedge sampling requires a minimum of 2 and a maximum of $2q$ edges. If $2q > d$, some edges are necessarily reused. If $2q \ll d$, it is more likely that every wedge centered at v will have two unique edges; however, due to the birthday paradox, there remains a nonnegligible likelihood of wedge overlap even for large d. For these large d, we want to avoid reading all neighbors into memory since the list is quite long, but we still want to accurately reproduce uniform sampling with replacement. We do this by using a simulated sampling procedure explained below. It only requires reading the first d' neighbors into memory where $d' \leq \min\{ d, 2q \}$.

Procedure Sampling produces q uniform random wedges (with replacement) centered at v. Number the edges incident to v arbitrarily from 1 to d. A uniform random wedge is represented as a uniform random pair of indices (i, j) $(i \neq j, i \leq d, j \leq d)$. We can repeat this random index selection q times to implicitly sample q random wedges, each of which is just represented as a pair of indices. Observe that the total number of indices in the union of these pairs is at most d', so all we need are the first d' uniform randomly ordered edges obtained as the output of the Map phase. We map these sampled wedge indices randomly to the index set $\{1, 2, \ldots, d'\}$ through a permutation. Now, each wedge is indexed as a pair (i, j) $(i \neq j, i \leq d', j \leq d')$. From the list of edges/neighbors $\{x_1, x_2, \ldots, x_{d'}\}$, we can generate these random edges. This is what is done in Sampling and Reduce in Algorithm 4.

The final output of this phase is a *sample wedge list file*, where each entry is of the form $(h, v_0, v_1, v_2, p, d_0)$. The number h is a hash of the desired closure edge (v_1, v_2) (a key which allows the undirected edges from the edge list to be correctly matched with the closure requests in Phase 2b), the wedge is defined by (v_1, v_0, v_2), p is the total number of wedges in the bin containing v_0, and d_0 is the degree of vertex v_0.

As mentioned above, Phase 2b is optional. If skipped, the MapReduce shuffle brings adjacent edges of a wedge center together in the reduce phase. We defer calculation of the number of sample wedges to the reduce phase but otherwise proceed as defined above. Note that in many cases the reducer collects zero samples and does no work.

4.5. Phase 3: Check sample wedge closure.

4.5.1. Phase 3a: Gather sample wedge closure hashes. Phase 3a (optional) assembles a list of all the unique edge hashes from the *sample wedges file* and stores it as a Java `Set` object. We denote this *wedge hashes object* by $\xi = \{ h_1, h_2, \ldots \}$. This is similar to the procedure in Phase 2b, which assembles the list of wedge centers. We set $\xi = \emptyset$ if Phase 3a is skipped.

Algorithm 4. Create sample wedges (Phase 2c).

```
parameters: $\tau, \omega$                                   ▷ Binning parameters
parameter: $k$                                   ▷ Desired number of samples per bin
parameter: $\gamma$                                   ▷ Represents wedge centers object

method Map($v, d, q, p$)                                   ▷ Input is wedge centers file
    Emit($v$:0, $(d, q, p)$)                                   ▷ Note secondary sort key

method Map($v, w$)                                   ▷ Input is edge list file
    EdgeHelper($v, w$)
    EdgeHelper($w, v$)

method EdgeHelper($v, w$)
    $b \leftarrow \gamma(v)$                                   ▷ Extract bin ID
    if $b = 0$ then                                   ▷ Phase 2b was skipped
        $\phi \leftarrow 1$                                   ▷ Always emit the edge
    else if $b = 1$ then                                   ▷ Vertex is not a wedge center
        $\phi \leftarrow 0$                                   ▷ Never emit the edge
    else                                   ▷ Vertex is a wedge center
        $d_{\min} =$ BinLoDeg($b, \tau, \omega$)                                   ▷ Lower bound degree of $v$
        $\phi \leftarrow 2 \cdot (2k/d_{\min})$                                   ▷ Proportion of edges to emit for $v$
    end if
    $x \leftarrow$ Rand($[0, 1]$)                                   ▷ Uniform random number in $[0, 1]$
    if $x \leq \phi$ then                                   ▷ Probabilisticly downselect
        $y \leftarrow$ Rand($\{ 1, \ldots, \texttt{maxlongint} \}$)                                   ▷ Random long integer
        Emit($v$:$y$, $w$)                                   ▷ Note secondary sort key
    end if

method Reduce($v, \{ x_1, x_2, \ldots \}$)
    if $x_1$ is a wedge center then ▷ If it exists, the wedge center information is first
        $(d, q, p) \leftarrow x_1$                                   ▷ Unpack wedge center information
        $(d', \{ (i_\ell, j_\ell) \}_{\ell=1}^{q}) \leftarrow$ Sampling($d, q$)                                   ▷ Determine sample wedges
        $\{ w_1, \ldots, w_{d'} \} \leftarrow \{ x_2, x_3, \ldots, x_{d'+1} \}$                                   ▷ Read only $d'$ neighbors
        for each $\ell = 1, \ldots, q$ do
            $h \leftarrow$ Hash($w_{i_q}, w_{j_q}$)                                   ▷ Hash of edge that would close this wedge
            Emit($h, v, w_{i_q}, w_{j_q}, p, d$)                                   ▷ Output is sample wedges file
        end for
    end if

method Sampling($d, q$)                                   ▷ Subroutine for simulated sampling
    for each $\ell = 1, \ldots, q$ do                                   ▷ Generate endpoints for each wedge
        $i_\ell \leftarrow$ Rand $\{ 1, \ldots, d \}$
        $j_\ell \leftarrow$ Rand $\{ 1, \ldots, d \} \setminus \{ i_\ell \}$
    end for
    $\mathcal{S} \leftarrow \{ i_1, \ldots, i_q \} \cup \{ j_1, \ldots, j_q \}$   ▷ Gather unique indices (duplicates removed)
    $d' \leftarrow |\mathcal{S}|$                                   ▷ Number of edges needed
    Define mapping $\pi : \mathcal{S} \to \{ 1, \ldots, d' \}$                                   ▷ Renumber from 1 to $d'$
    return $d'$ and pairs $\{ (\pi(i_\ell), \pi(j_\ell)) \}_{\ell=1}^{q}$
```

Algorithm 5. Check sample wedge closure (Phase 3b).

```
parameter: $\xi = \{ h_1, h_2, \dots \}$                    ▷ Represents wedge hashes object

method Map($h, v_0, v_1, v_2, p, d_0$)                      ▷ Input is sample wedges file
    Emit($h$, ($v_0, v_1, v_2, p, d_0$))

method Map($w_1, w_2$)                                      ▷ Input is edge list file
    $h \leftarrow$ Hash($w_1, w_2$)                          ▷ Hash of edge
    if ($\xi = \emptyset$) or ($h \in \xi$) then
        Emit($h$, ($w_1, w_2$))
    end if

method Reduce($h, \{ x_1, x_2, \dots \}$)
    Sort the values $\{ x_1, x_2, \dots \}$ into $\mathcal{E}$ (edges) and $\mathcal{W}$ (wedges)
    for each $w \in \mathcal{W}$ do
        $(v_0, v_1, v_2, p, d_0) \leftarrow w$               ▷ Unpack wedge data
        $\sigma \leftarrow$ "open"                           ▷ By default, wedges are open
        for each $e \in \mathcal{E}$ do
            $(w_1, w_2) \leftarrow e$                        ▷ Unpack edge data
            if ($w_1 = v_1$ and $w_2 = v_2$) or ($w_2 = v_1$ and $w_1 = v_2$) then
                $\sigma \leftarrow$ "closed"
            end if
        end for
        Emit($\sigma, v_0, v_1, v_2, p, d_0$)                ▷ Output is results file (ver. 0)
    end for
```

4.5.2. Phase 3b: Check sample wedge closure. Phase 3b is the last major step and checks the wedge closures, as shown in Algorithm 5. The inputs are the *sample wedges file* created by Phase 2c and the original *edge list file*. We also pass the optional *wedge hashes object* (ξ) as a configuration parameter. If ξ is nonempty, it is used to filter the edges passed forward to the reduce function. (Note that we could skip Phase 3a and forward every edge to the reducers, but this would result in much greater data shuffling in Phase 3b.) Note that more than one edge may hash to the same value; hence, we loop through all edges that arrive at the reducer to verify that there is a match before declaring a wedge as closed. Likewise, more than one wedge may be closed by a single edge. The output of this phase is the *results file (ver. 0)*; each entry is of the form $(\sigma, v_0, v_1, v_2, p, d_0)$, where σ indicates if the wedge is open or closed and everything else is the same as for the *sample wedges file*.

4.6. Phase 4: Postprocessing.

4.6.1. Phases 4a and 4b: Find degrees of wedge endpoints. Phases 4a and 4b augment each sample wedge with the degrees of v_1 and v_2. This information is needed for estimating the number of triangles per bin. If only the clustering coefficients are required, these two steps can be omitted. Algorithm 6 shows Phase 4a; the procedure for Phase 4b is analogous and so is omitted. The final output of Phase 4b is the *results file (ver. 2)*; each line is of the form $(\sigma, v_1, v_o, v_2, p, d_0, d_1, d_2)$, where d_1 and d_2 are the degrees of vertices v_1 and v_2, respectively, while the remainder is the same as for the *results file (ver. 0)*.

ALGORITHM 6. Find degree of first vertex per wedge (Phase 4a).

```
method MAP($\sigma, v_0, v_1, v_2, p, d_0$)                ▷ Input is results file (ver. 0)
    EMIT($v_1$:1, ($\sigma, v_0, v_1, v_2, p, d_0$))

method MAP($v, d$)                                          ▷ Input is vertex degree file
    EMIT($v$:0, $d$)

method REDUCE($v, \{ x_1, x_2, \dots \}$)                   ▷ Add the degree of $v_1$ for each wedge.
    $d_1 \leftarrow x_1$                                    ▷ First value is the degree of the vertex
    for each $x \in \{ x_2, x_3 \dots \}$ do                ▷ Remaining values, if any, comprise sample
wedges
        EMIT($x, d_1$)                                      ▷ Output is results file (ver. 1)
    end for
```

ALGORITHM 7. Summarize results (Phase 4c).

```
parameters: $\tau, \omega$                                  ▷ Binning parameters

method MAP($\sigma, v_0, v_1, v_2, p, d_0, d_1, d_2$)       ▷ Input is results file (ver. 2)
    $b_0 \leftarrow$ BINID($d_0, \tau, \omega$)
    $b_1 \leftarrow$ BINID($d_1, \tau, \omega$)
    $b_2 \leftarrow$ BINID($d_2, \tau, \omega$)
    EMIT($b$, ($\sigma, p, b_1, b_2$))

method REDUCE($b, \{ x_1, x_2, \dots \}$)
    $p_0, p_1, p_2, p_3 \leftarrow 0$
    for each $x \in \{ x_1, x_2 \dots \}$ do
        $(\sigma, p, b_1, b_2) \leftarrow x$                ▷ Unpack value
        if $\sigma =$ "open" then
            $q_0 \leftarrow q_0 + 1$
        else
            $i \leftarrow 1 + (b = b1) + (b = b2)$
            $q_i \leftarrow q_i + 1$
        end if
    end for
    $c \leftarrow (q_1 + q_2 + q_3)/(q_0 + q_1 + q_2 + q_3)$
    $t \leftarrow p \cdot (q_1 + q_2/2 + q_3/3)/(q_0 + q_1 + q_2 + q_3)$
    EMIT($b, q_0, q_1, q_2, q_3, c, p, t$)                  ▷ Output is summary file
```

4.6.2. Phase 4c: Summarize results. Phase 4c tallies the final results per bin, using the logic in Algorithm 7. Its output is the *summary file.* Each line is of the form $b, q_0, q_1, q_2, q_3, c, p, t$, where b is the bin ID, q_0 is the number of open wedges, q_i is the number of closed wedges with i vertices in the bin, c is the clustering coefficient estimate, p is the number of wedges in the bin, and t is the estimated number of triangles with one or more vertices in the bin.

We can estimate the global clustering coefficient from the degree-binned clustering coefficients as follows. Let $\hat{c}_b$ and p_b be the clustering coefficient estimate and total number of wedges for bin b. Let $p = \sum_b p_b$ be the total number of wedges. Then

Table 1
Input, shuffle, and output volumes for MapReduce phases.

Phase	Mapper input	Key-value pairs	Reducer output
1a	m: Process m edges from *edge list file*.	$2m$: Communicate 2 key-value pairs per edge.	n: Output n vertex-degree pairs to *vertex degrees file*.
1b	n: Process n vertex-degree pairs from *vertex degrees file*.	n: Communicate 1 key-value pair per vertex.	$b_{\max}$: Output data for each bin to *wedges per bin file*.
2a	n: Process n vertex-degree pairs from *vertex degrees file*.	q: Communicate/output approximately q sample wedge centers in *wedge centers file*.	
2c	$m+q$: Process m edges from *edge list file* and approximately q sample wedge centers from *wedge centers file*.	$O(qk)+q$: Communicate $O(k)+1$ key-value pair per sample wedge center.	q: Output the sample wedges to the *sample wedges file*.
3b	$q+m$: Process wedges in *sample wedges file* and edges from *edge list file*.	$2q$: Communicate 1 message per wedge and 1 message per hash-matching edge.	q: Output close/open data for each sample wedge into *results file (ver. 0)*.
4a/4b	$n+q$: Process q sampled wedges from *results file (ver. 0/1)* and n vertex-degree pairs to *vertex degrees file*.	$n+q$: Communicate 1 key-value pair for each vertex and each edge.	q: Output augmented data for each sample wedge into *results file (ver. 1/2)*.
4c	q: Process q sampled wedges from *results file (ver. 2)*.	$3q$: Communicate 3 key-values pairs per wedge.	$b_{\max}$: Output results per bin in *summary files*.

the estimates for the global clustering coefficient and total number of triangles are given by

$$\hat{c} \approx \sum_b \frac{p_b}{p} \cdot \hat{c}_b \quad \text{and} \quad \hat{t} = \hat{c} \cdot \frac{p}{3}. \tag{4.3}$$

Let $b_{\max}$ denote the total number of bins. We assume that every bin has k samples producing an error bound of ε with confidence $(1-\delta)$. Then we argue that $|c - \hat{c}| \leq \varepsilon$ with confidence $(1 - b_{\max} \cdot \delta)$.

4.7. Performance analysis. Table 1 presents the input, communication, and output volume for each phase. Let n denote the number of nodes, m denote the number of edges (each undirected edge is counted just once), q denote the total number of sampled wedges, and $b_{\max}$ denote the number of bins. Note that the communications in Phases 2c and 3b can be substantially higher $(m+q)$ if Phase 2b or 3a is skipped. Our experimental results show that Phase 1a is by far the most expensive, which is consistent with our performance analysis because Phase 1a communicates the most data, $2m$ key-values pairs. All other communications are size n or q.

5. Experimental results.

5.1. Data description. We obtained real-world graphs from the Laboratory for Web Algorithms (http://law.di.unimi.it/datasets.php), which were compressed using LLP and WebGraph [11, 9]. We selected 10 larger graphs for which the complete edge lists were available. We also consider three artificially generated graphs according to the Graph500 benchmark [67], which uses stochastic Kronecker graphs (SKG) [33] for its graph generator with [0.57,0.19;0.19,0.05] as the 2×2 generator

TABLE 2
Network characteristics. All edges are treated as undirected. The triangle counts and global clustering coefficients (GCC) are our estimates.

ID	**Graph name**	**Nodes (millions)**	**Edges (millions)**	**Wedges (millions)**	**Triangles (millions)**	**GCC**
1	amazon-2008	1	4	51	4	0.2603
2	ljournal-2008	5	50	9,960	408	0.1228
3	hollywood-2009	1	56	47,645	4,907	0.3090
4	hollywood-2011	2	114	120,899	7,097	0.1761
5	graph500-23	5	128	567,218	3,673	0.0194
6	it-2004	41	1,027	16,163,308	48,788	0.0091
7	graph500-26	34	1,054	9,087,164	28,186	0.0093
8	twitter-2010	42	1,203	123,435,590	34,495	0.0008
9	uk-2006-06	80	2,251	16,802,569	186,453	0.0333
10	sk-2005	43	2,543	5,196,166,169	256,556	0.0001
11	uk-2006-05	77	2,636	167,591,218	363,111	0.0065
12	uk-union	132	4,663	203,567,548	447,133	0.0066
13	graph500-29	240	8,502	158,727,767	272,931	0.0052

matrix. We have added noise with a parameter of 0.1, as proposed in [52, 53] to avoid oscillatory degree distributions. These graphs are generated in MapReduce. All networks are treated as undirected for our study; in other words, if $x \to y$, $y \to x$, or both, we say that edge (x, y) exists. Briefly, the networks are described as follows:

- amazon-2008 [11, 9]: A graph describing similarity among books as reported by the Amazon store.
- ljournal-2008 [14, 11, 9]: Nodes represent users on LiveJournal. Node x connects to node y if x registered y as a friend.
- hollywood-2009, hollywood-2011 [11, 9]: This is a graph of actors. Two actors are joined by an edge whenever they appear in a movie together.
- twitter-2010 [30, 11, 9]: Nodes are Twitter users, and node x links to node y if y follows x.
- it-2004 [11, 9]: Links between Web pages on the .it domain, provided by IIT.
- uk-2005-05, uk-2006-06, uk-union-2006-06-2007-05 (shorted to uk-union) [10, 11, 9]: Links between Web pages on the .uk domain. (We ignore the time labeling on the links in the last graph.)
- sk-2005 [8, 11, 9]: Links between Web pages on the .sk domain.
- graph500-23, graph500-26, graph500-29 [67, 33, 52, 46]: Artificially generated graphs according to the Graph500 benchmark using the SKG method. The number (23, 26, 29) indicates the number of levels of recursion and the size of the graph.

The properties of the networks are summarized in Table 2; specifically, we report the number of vertices, the number of *undirected* edges, the total number of wedges, and estimates for the total number of triangles and the global clustering coefficients, calculated according to (4.3). (To the best of our knowledge, we are the only group that has calculated the last three columns, so these numbers have not been independently validated.)

5.2. Experimental setup. We have conducted our experiments on a Hadoop cluster with 32 compute nodes. Each compute node has an Intel i7 930 CPU at 2.8GHz (four physical cores, HyperThreading enabled), 12 GB of memory, and four

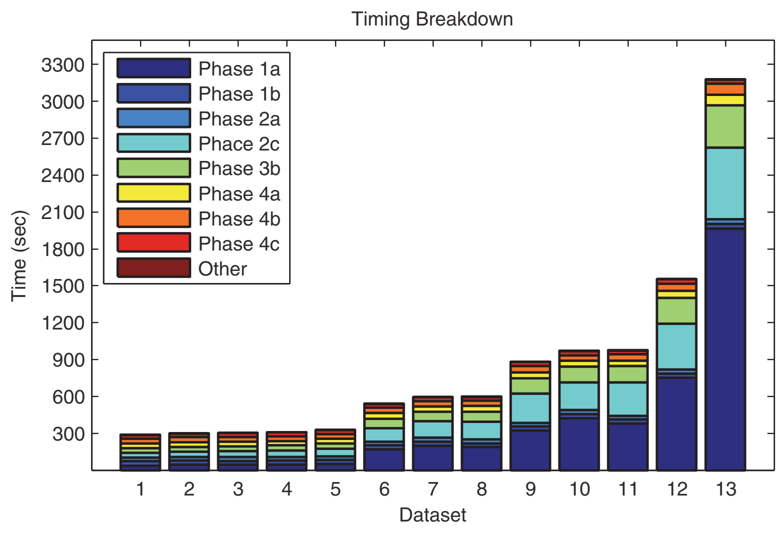


FIG. 3. *Runtimes broken down by phases.*

2TB SATA disks. All experiments were run using Hadoop version 0.20.203.0. Unless otherwise stated, all experiments use the following parameters:

- Number of samples per bin: $k = 10,000$;
- Bin parameters: $\tau = 2$ and $\omega = 2$ (i.e., bins are $\{2\}, \{3,4\}, \{5,6,7,8\}, \ldots$);
- Number of reducers: 64.

5.3. Experimental results and timings. We ran our MapReduce code on the 13 networks described in section 5.1. The runtimes are reported in Figure 3, broken down by the phases of the algorithm. The largest real-world graph, uk-union (#12) with over 100 million vertices and over 4.6 billion edges, took less than 30 minutes to analyze. For all the networks, the most expensive step is Phase 1a, calculating the degree per node, because every edge in the *edge list file* generates two key-value pairs. For uk-union, this step takes over 12 minutes. Phases 1b and 2a are essentially constant time (approximately 30 seconds) because they process only the *vertex degree file*. Phase 2c (create sample wedges) is the next most expensive step. Here we collect the edges adjacent to wedge centers, reading the entire *edge list file* again in the map phase; however, the *wedge centers file* created in Phase 2b minimizes the number of edges that are passed to reducers. Nevertheless, for the wedge centers with high degree, many edges are transmitted (though substantially less than the entire edge list). Phase 3b (check sample wedge closure) is the next most expensive, again reading the entire *edge list file*. The last few postprocessing steps are close to constant time (approximately 30–60 seconds each).

In an enumeration approach, we expect the runtimes to be proportional to the number of wedges. In that case, the sk-2005 (#10) would be the most expensive by an order of magnitude. For our method, however, the runtime is proportional to the number of edges. Figure 4 shows that there is a near-linear relationship between the number of edges and the runtime. The x-axis is the number of edges (in millions) and the y-axis is the runtime. All 13 examples are included. We see that there is a constant cost of 225 seconds, which accounts for the MapReduce overhead and then an incremental cost of 0.33 seconds per million edges.

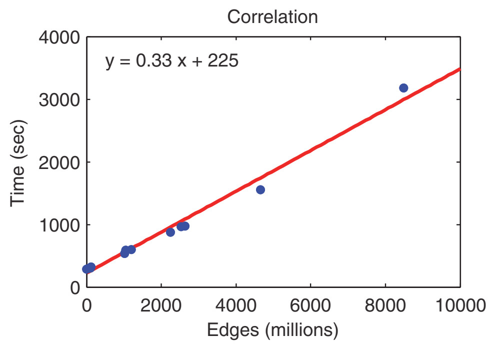


FIG. 4. *Correlation of the runtime to the number of edges. There is a setup cost of* 225 *seconds due to the MapReduce overhead and an incremental cost of* 0.33 *seconds per million edges.*

TABLE 3
Comparison of sampling and exact enumeration.

Id	**Graph name**	**Wedges checked**	**Runtime (sec.)**	**Exact GCC**	**Sampling error**	**Sampling speed-up**
1	amazon-2008	25%	158	0.2603	0.0001	1
2	ljournal-2008	19%	3,385	0.1228	0.0010	11
3	hollywood-2009	29%	21,665	0.3090	0.0006	72
4	hollywood-2011	27%	90,598	0.1761	0.0006	293

Comparisons to other methods. As one point of comparison, we ran Plantenga's implementation that fully enumerates triangles [47] on several smaller graphs. The results are summarized in Table 3. This implementation is efficient because it only checks wedges where the center degree is smallest, i.e., (u, v, w) such that $d_v \leq d_u$ and $d_v \leq d_w$. This means that only one wedge per triangle is checked and moreover that many open wedges centered at high-degree nodes are ignored. The table lists the percentage of wedges that are checked for closure. The run times range from 3 minutes for the smallest graph up to 25 hours for hollywood-2011 (2M nodes, 114M edges). Because this code enumerates every triangle, we can calculate the exact GCC. The error from our sampling method is also reported. At a confidence level of 99.9%, using $k = 2000$ samples yields an error of $\varepsilon = 0.05$. The true errors are one to two orders of magnitude less than this worst-case probabilistic bound. Finally, we observe the main advantage of sampling in terms of the observed speed-up, up to 293X for hollywood-2011. Larger graphs cannot be completed in a reasonable amount of time on our cluster.

Suri and Vassilvitskii [54] have an efficient enumeration method that was able to process two of the same data sets on a 1636-node Hadoop cluster: (a) ljournal-2008 required 5.33 minutes, and (b) twitter-2010 took 483 minutes. (In subsequent work, Arifuzzaman, Khan, and Marathe [3, 2] have reduced the runtime, and Park and Chung [43] were able to reduce the MapReduce run time to 213 minutes on 47 nodes.) On our 32-node cluster, our method requires approximately the same running time for the smaller data set (since most of the work is overhead) but requires only 10 minutes for twitter-2010.

We also compare to in-memory methods. We used an SGI Altix UV 10 System with 4 Xeon 8-core nodes and 64 $\times$ 8GB memory, for a total of 32 cores and 512GB of memory. We consider two methods for exact node-level triangle counting on the twitter-2010 graph: (1) MTGL [6] using the algorithm described in [16] and

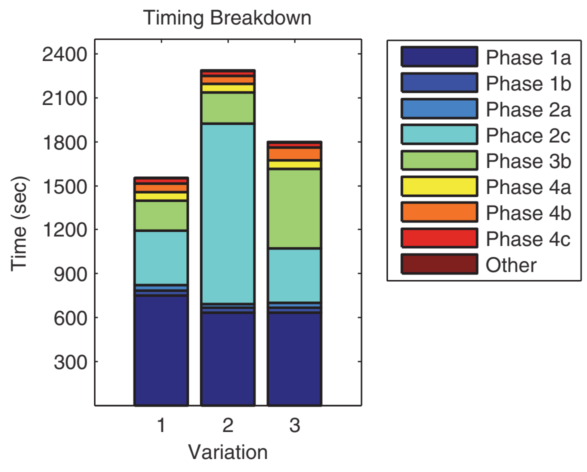


FIG. 5. *Timings for variations. The variations are* (1) *original algorithm,* (2) *skip Phase* 2*b, and* (3) *skip Phase* 3*a.*

(2) GraphLab/PowerGraph [37, 21]. On 32 cores, MTGL takes approximately 3.5 hours and consumes 125GB of memory. On 16 cores, GraphLab/PowerGraph takes approximately 17 minutes to run and 430GB of memory at its peak usage, nearly maxing out the machine's total memory. In both cases, just loading the graph takes 8–15 minutes. Compare these times to a total of 10 minutes using Hadoop and our inexact triangle-counting method.

The other class of methods for comparison are edge-sampling methods such as Doulion [59]. The basic idea is to sample a subset of *edges* and then run a triangle counting method (such as enumeration) on the reduced graph. Edge sampling has been compared to wedge sampling in serial [53]. Keeping 1 in 25 edges produces results that are roughly comparable to wedge sampling in time but with much greater variance in the GCC estimate. Keeping fewer edges yields savings in time but at the expense of much greater variance. Hence, we have not compared parallel implementations.

Impact of implementation features. We have considered many alternatives during the implementation of the wedge sampling algorithm, and in this subsection we present the impact of two implementation features. The three versions of the code we compare are (1) original algorithm, (2) skip Phase 2b, and (3) skip Phase 3a. We show results for uk-union in Figure 5. Skipping Phase 2b means that every edge generates two messages in Phase 2c, increasing the time in that phase from 372 seconds to 1235 seconds (3X increase), twice as expensive as Phase 1a. Skipping Phase 3a means that every edge generates one message in Phase 3b, increasing the time in that phase from 207 seconds to 543 seconds (2.5X increase). Hence, taking measures to reduce the data that must be shuffled to the reducers has major payoffs in terms of performance.

5.4. Degree distribution. The output of Phase 1b yields the (binned) degree distribution. We show results for 12 networks in Figure 6. (We omit uk-2006-05 because it is similar to uk-2006-06.) For each data point, the x-coordinate is the minimum degree of the bin, and the y-coordinate is the total number of vertices in that bin.

The degree distributions can be roughly characterized as heavy-tailed. None of the real-world graphs are particularly smooth in the degree distribution, and some

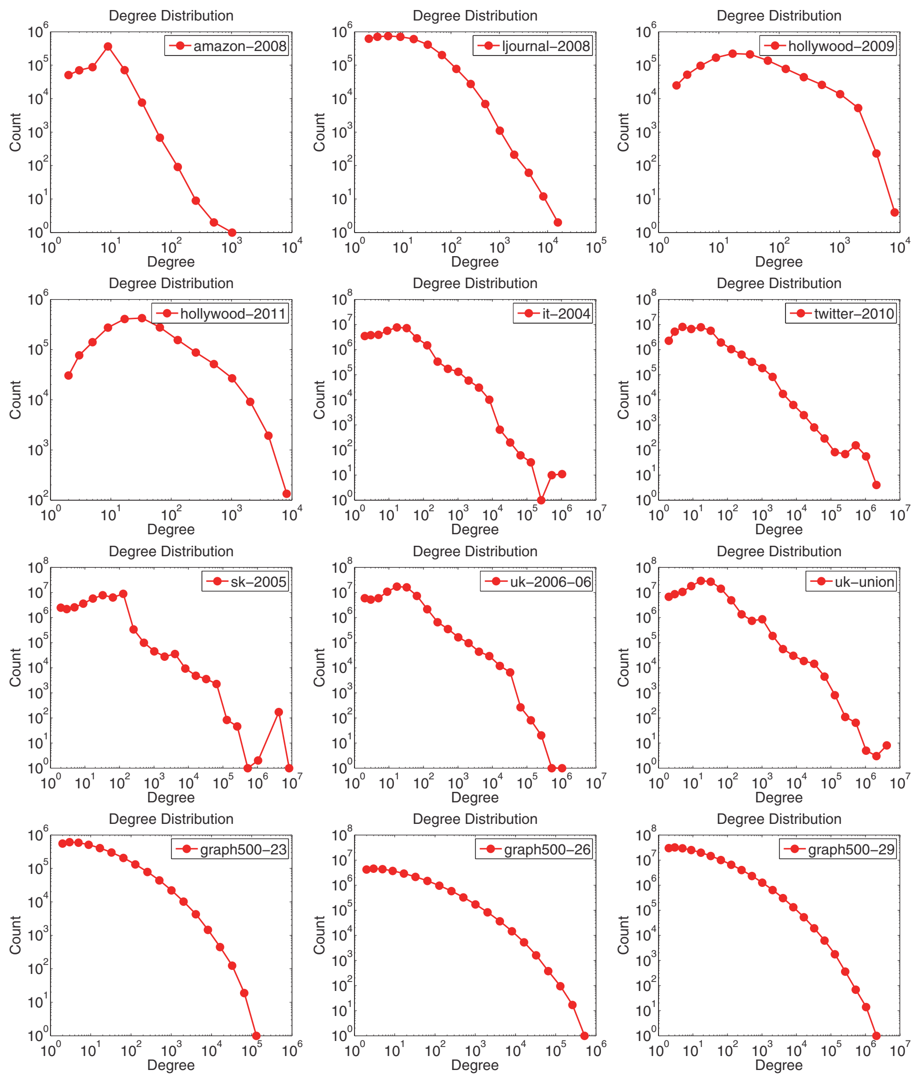


FIG. 6. *Degree distributions by bin.*

have odd spikes, especially in the tails (e.g., sk-2005, it-2004, uk-union). The artificial graphs (graph500-23/26/29) are extremely smooth; we know from analysis that the noisy version of Graph500 yields lognormal tails [52, 53].

5.5. Clustering coefficients. Clustering coefficients for each bin are displayed in Figure 7. Here the x-coordinate is the minimum degree in the bin, and the y-coordinate is the average clustering coefficient for wedges with centers in that bin.

Social networks are well known to have not only high clustering coefficients but also clustering coefficients that tend to degrade as the degree increases. This can be seen in the following graphs: amazon-2008, ljournal-2008, hollywood-2009, hollywood-2011. The twitter-2010 graph is not as "social" in terms of clustering coefficient.

The web graphs (it-2004, uk-2006-06, uk-union, sk-2005) are interesting because the clustering coefficients seems to start low, increase, and then drop off quickly. This

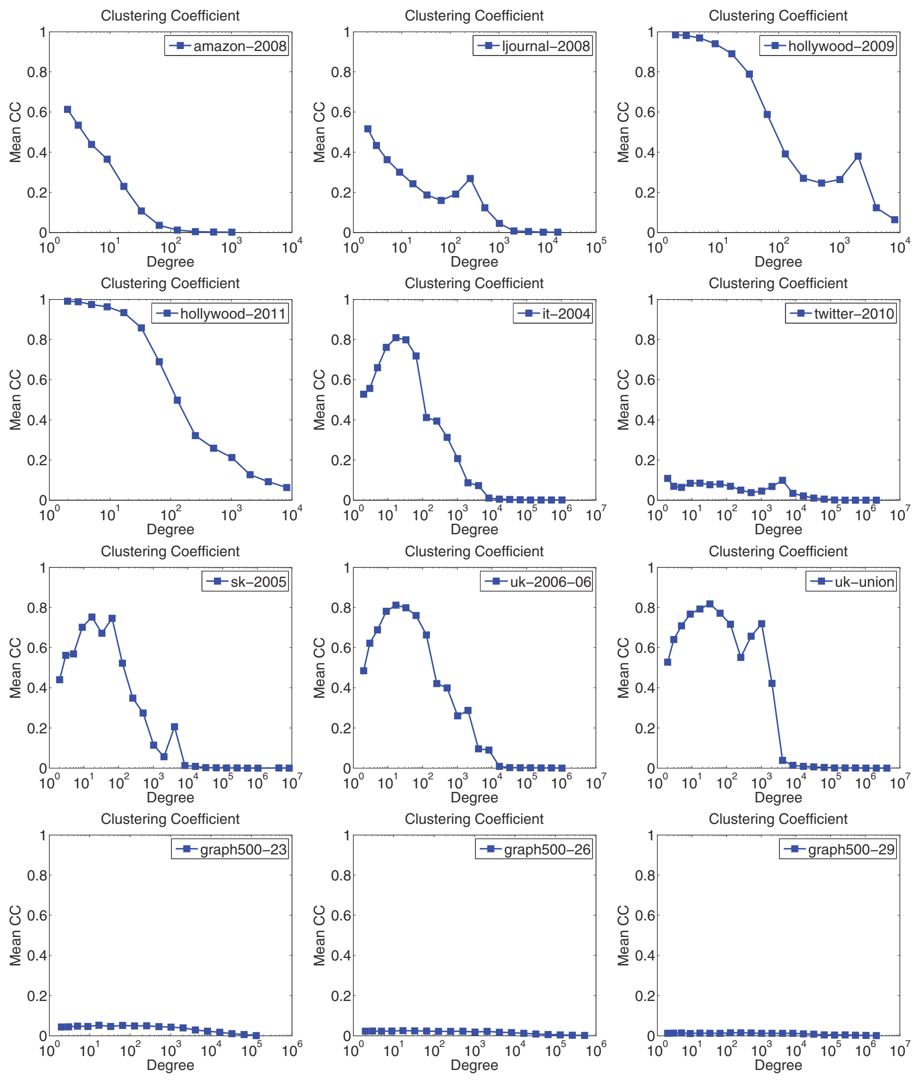


FIG. 7. *Clustering coefficients by bin.*

may be due to the design of Web sites with many interconnected pages or to some artifact of the crawling process.

The Graph500 examples have overall low clustering coefficients and do not behave like the real-world graphs. The closest match is the twitter-2010 graph.

5.6. Triangles. We also present the number of triangles per bin in Figure 8. Here the x-coordinate is the minimum degree in the bin, and the y-coordinate is the proportion of triangles that have at least one vertex in that bin. Triangles may be counted more than once if they have different vertices in different bins.

It is interesting to observe where triangles come from. Even though low-degree nodes are the most plentiful, most of the triangles come from higher-degree nodes. We can roughly sort the graphs into three categories.

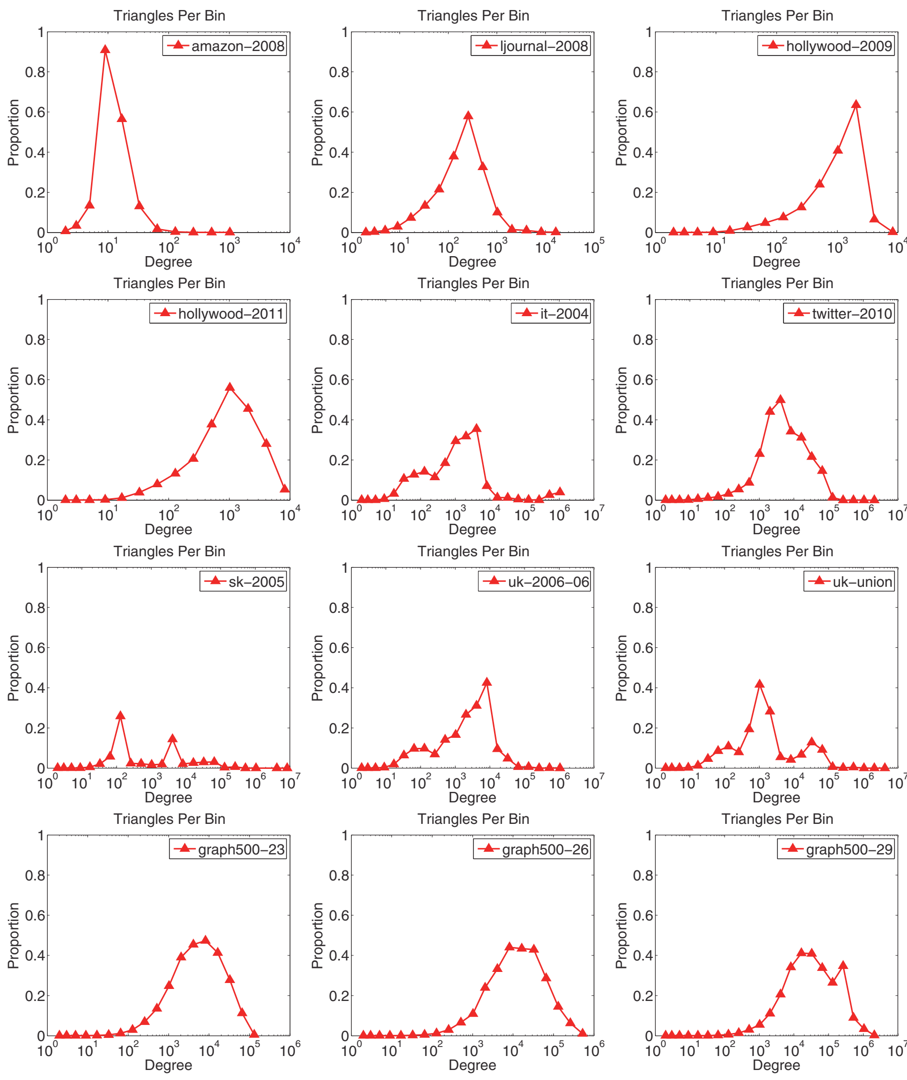


Fig. 8. *Proportion of triangles in each bin.*

There is only one graph where the triangles come from relatively low-degree vertices: amazon-2008. Here it seems that most triangles come from nodes with degrees between 4 and 80.

There are several graphs where the triangles come from the midrange degrees: one social graph (ljournal-2008) and all the Web graphs (it-2004, sk-2005, uk-2006-06, uk-union). The "double-spike" behavior of sk-2005 is interesting.

Finally, there are a few graphs where the vast majority of triangles involve the high-degree nodes. Both Hollywood graphs are of this type; note that 60% of the triangles involve one of the nodes in the bin starting at degree 1025. The Graph500 graphs also have most of the triangles coming from the highest-degree nodes.

5.7. Triangle statistics. An interesting feature of our wedge sampling techniques is that in the case of a single bin, all the closed triangles are uniformly

Table 4
Number of triangles from 5,000,000 *wedge samples.*

Graph name	**Time(s)**	**Triangles**
uk-union	2618	33398
hollywood-2011	348	878719
graph500-26	845	46047
graph500-29	5487	25994

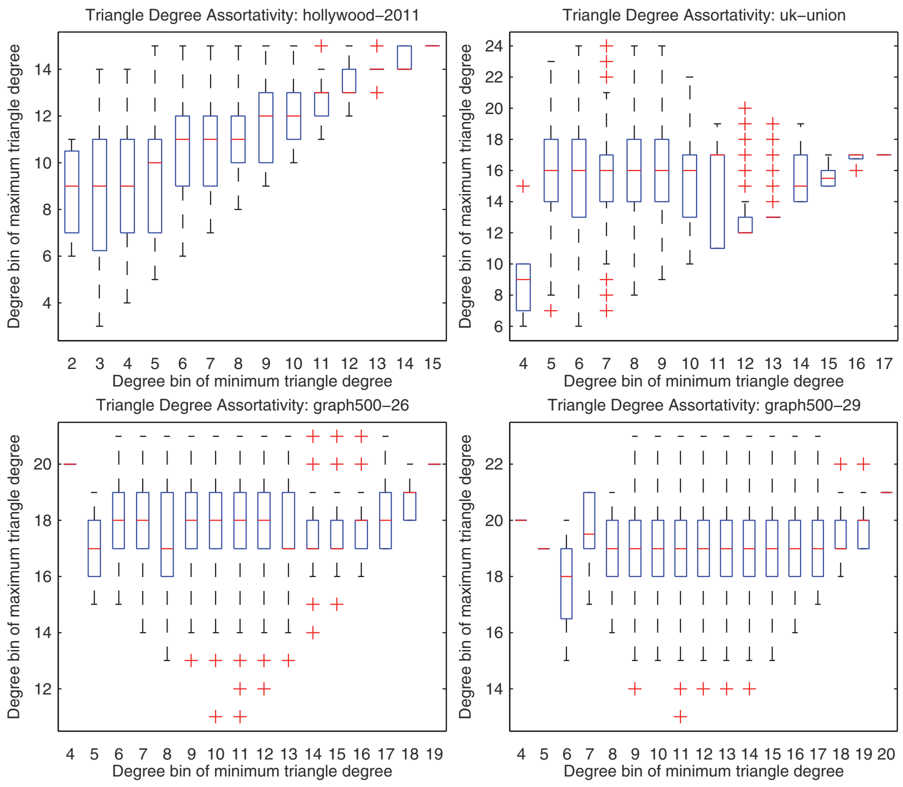


Fig. 9. *Triangle degree assortativity.*

randomly sampled as well. Such a random sample can be used to analyze the characteristics of the triangles in the graph, going further than merely looking at their count. Examples of such studies can be found in [19], where full enumeration of the triangles was used. To avoid the burden of full enumeration a uniform sampling of the triangles can be used, as we showcase below.

For four example graphs, we ran our MapReduce code with a single bin ($\tau = 1$ and $\omega = 10^7$) and $k = 5,000,000$ samples; we skipped Phases 2b and 3a to avoid any data overflow problems in the configuration parameters. Runtimes and the number of triangles (expected to be roughly k times the global clustering coefficient) are reported in Table 4.

Using these sampled triangles, we can look at the degrees of the vertices. Each triangle has a minimum, middle, and maximum degree. We analyze the *degree assortativity* of the vertices of the triangles by comparing the minimum and maximum degrees in Figure 9. Specifically, we assign each vertex to a degree bin, using (4.2) with $\tau = 2$ and $\omega = 2$. We group together all triangles with the same minimum degree bin. The box plot shows the statistics of the bin for the maximum degree: the central mark (red) is the median max-degree, while the edges of the (blue) box are the 25th and 75th percentiles. The whiskers extend to the most extreme points considered not to be outliers, and the outliers (red plus marks) are plotted individually.

Observe that the social network, hollywood-2011, shows an assortative relation between the maximum and minimum for the hollywood graph, since the two quantities rise gradually together. For the uk-union Web graph, on the other hand, the average maximum degree is essentially invariant to the minimum degree. These findings are consistent with the results in [19] about networks with high global clustering coefficients having degree assortative triangles, while this assortativity cannot be observed in networks with low clustering coefficients. Here, we were able to observe the same trend on these massive graphs using sampling in a much more efficient way, avoiding the enumeration burden. We see that the Graph500 networks also have almost no assortativity between the minimum and maximum degrees and therefore do not have the characteristics of a social network.

6. Conclusions.

We have shown that wedge-based sampling can be scaled to massive graphs in the MapReduce framework. On a relatively small MapReduce cluster (32 nodes), we have analyzed graphs with up to 240M edges, 8.5B edges, 5.2T wedges, and 447B triangles. Even the largest graph was analyzed in less than one hour, and most took only a few minutes. Figure 10 shows a timing analysis of the MapReduce tasks [65] for Phase 1a on the uk-union graph. Mapper tasks run in waves of 128 parallel jobs, equal to the number of mapper slots available on the cluster. Note that a larger cluster would be able to run more Map tasks in parallel, decreasing the overall runtime. To the best of our knowledge, these are the largest triangle-based calculations performed to date.

Unlike enumeration techniques that need to at least validate every triangle and more often have cost proportional to the number of wedges, our method is linear in the number of edges. The most expensive component of the wedge-based sampling is finding the degree of each vertex (Phase 1a); reducing the time for this is a topic for future study. On our cluster, the time is approximately 0.33 seconds per million edges, plus a fixed cost of 225 seconds for overhead. Because we are using MapReduce, we never need to fit the entire graph into memory—we only need to be able to stream through all the edges.

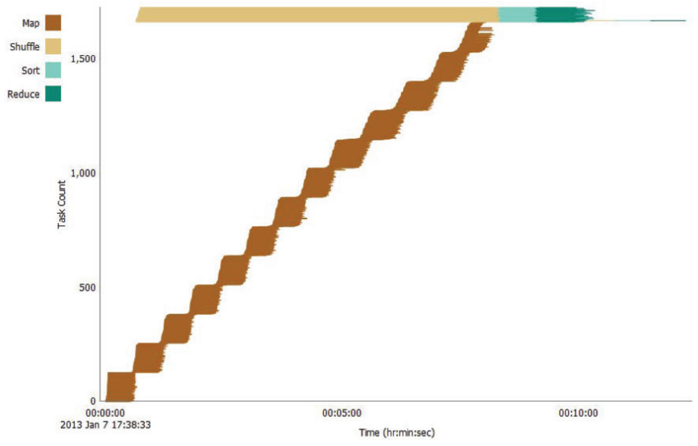


FIG. 10. *Task breakdown for Phase 1a on uk-union on 32 Hadoop nodes.*

Our MapReduce implementation requires a total of eight MapReduce jobs, three of which do most of the work because they read the entire edge list (Phases 1a, 2c, and 3b) and two of which are optional (Phases 4a and 4b, which are labeling the degrees of the sampled triangles). We have striven to minimize the data being shuffled in each phase by using special data structures to filter the edges.

Using our code, we are able to compute the degree distribution and approximate the binned degreewise clustering coefficient and the number of triangles per bin. Additionally, we can analyze the characteristics of the triangles (e.g., degree assortativity). As part of our analysis, we have analyzed the graphs used in the Graph500 benchmark. We are able to give a more detailed understanding of the empirical properties of the generator and compare it to real-world data; this is potentially helpful in determining if performance on the benchmark data is indicative of performance on real-world data.

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